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Constructibility of the Set of Polynomials with a Fixed Bernstein–Sato Polynomial: an Algorithmic Approach

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Let n and d be positive integers, let k be a field and let $\mathcal{P}(n, d; k)$ be the space of the non-zero polynomials in n variables of degree at most d with coefficients in k. Let $B(n, d)$ be the set of the Bernstein–Sato polynomials of all polynomials in $\mathcal{P}(n, d; k)$ as k varies over all fields of characteristic 0. G. Lyubeznik proved that $B(n, d)$ is a finite set and asked if, for a fixed k, the set of the polynomials corresponding to each element of $B(n, d)$ is a constructible subset of $\mathcal{P}(n, d; k)$.

In this paper we give an affirmative answer to Lyubeznik's question by showing that the set in question is indeed constructible and defined over $\mathbb{Q}$, i.e. its defining equations are the same for all fields k. Moreover, we construct an algorithm that for each pair (n, d) produces a complete list of the elements of $B(n, d)$ and, for each element of this list, an explicit description of the constructible set of polynomials having this particular Bernstein–Sato polynomial.

1. Introduction

Throughout this paper k is a field of characteristic 0, $R_n(k) = k[x_1, \ldots, x_n]$ is the ring of polynomials in n variables and $A_n(k) = k\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ is the corresponding Weyl algebra, i.e. an associative k-algebra generated by x's and ∂'s with the relations $\partial_i x_i = x_i \partial_i + 1$ for all i.

For every polynomial $f \in R_n(k)$ there are $b(s) \in k[s]$ and $Q(x, \partial, s) \in A_n(k)[s]$ such that

$$b(s)f^s = Q(x, \partial, s) \cdot f^{s+1}. \tag{1}$$

For the proof of existence of such $b(s) \neq 0$ (see, Björk, 1979, for example). The polynomials $b(s)$ for which equation (1) exists form an ideal in $k[s]$. The monic generator of this ideal is denoted by $b_f(s)$ and called the Bernstein–Sato polynomial of f, which was first introduced by Bernstein in Bernstein (1972). A good introduction to the D-modules theory, may be found in Björk (1979).

The simplest characteristics of a polynomial f are its degree d and its number of variables n. This paper is motivated by the following natural question: what can one say about $b_f(s)$ in terms of n and d? We give what may be regarded as a complete answer to this question. Namely, we describe an algorithm that for fixed n and d gives a complete list of all possible Bernstein–Sato polynomials and, for each polynomial $b(s)$ in this list, a complete description of the polynomials f such that $b_f(s) = b(s)$.

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Let $\mathcal{P}(n, d; k)$ be the set of all the non-zero polynomials of degree at most d in n variables with coefficients in k and let $P(n, d; k)$ be $\mathcal{P}(n, d; k)$ modulo the equivalence relation $f \sim g \Leftrightarrow f = c \cdot g$ for some $0 \neq c \in k$. Note that $b_f(s) = b_g(s)$ if $f \sim g$. We view $P(n, d; k)$ as the set of the k-rational points of the projective space $\mathbb{P}(n, d; k) \cong \mathbb{P}_k^{N-1}$ where N is the number of monomials in n variables of degree at most d. Lyubeznik (1997) defined $B(n, d)$ as the set of all the Bernstein–Sato polynomials of all the polynomials from $\mathcal{P}(n, d; k)$ as k varies over all fields of characteristic 0 and he proved that $B(n, d)$ is a finite set. He also asked if the subset of $\mathcal{P}(n, d; k)$ corresponding to a given element of $B(n, d)$ is constructible. In this paper we show that the corresponding subset of $\mathbb{P}(n, d; k)$ is indeed constructible thus giving an affirmative answer to Lyubeznik's question. The constructible sets in question turn out to be definable over $\mathbb{Q}$, i.e. their defining equations and inequalities are the same for all fields k.

A crucial ingredient in our proof is the fact, very recently discovered by Oaku (1997) that there is an algorithm that, given a polynomial f, returns its Bernstein–Sato polynomial $b_f(s)$. Using Oaku's algorithm we have developed an algorithm that computes the complete set of the Bernstein–Sato polynomials $B(n, d)$ for each pair (n, d) and for each $b(s) \in B(n, d)$ constructs a finite number of locally closed sets $V_i = V_i' \setminus V_i''$, where V_i' and V_i'' are Zariski closed subsets of $\mathbb{P}(n, d; \mathbb{Q})$ defined by explicit polynomial equations with rational coefficients, such that for every field k of characteristic 0, the subset of $P(n, d; k)$ having $b(s)$ as the Bernstein–Sato polynomial is the set of k-rational points of

$$S(b(s), k) = (\cup_i V_i) \otimes_{\mathrm{Spec}\mathbb{Q}} \mathrm{Spec} k \subset \mathbb{P}(n, d; \mathbb{Q}) \otimes_{\mathrm{Spec}\mathbb{Q}} \mathrm{Spec} k = \mathbb{P}(n, d; k).$$

A similar approach applies also to computing the Bernstein–Sato polynomial of a polynomial with parameters. Namely, one can prove that the number of different possible Bernstein–Sato polynomials in this case is finite and the stratum for each of them is a constructible set in the space of parameters.

Moreover, using a similar technique we develop an algorithm for computing the annihilator of $\frac{1}{f^s}$ in $A_n(k)$, which, provided such s is known that $\frac{1}{f^s}$ generates $R_n(k)_f$, gives a presentation of $R_n(k)_f$ as an $A_n(k)$-module (see Example 4.5). This algorithm is particularly important for U. Walther's algorithmic computation of local cohomology modules (Walther, 1999).

These applications are discussed in Section 4.

It should be mentioned that the question of the constructibility of the stratification by Bernstein–Sato polynomials was treated in the analytic setup in case of deformations of hypersurfaces with isolated singularities (Briançon *et al.*, 1992, Theorem 3.3).

The results of this paper are a part of my thesis. I would like to thank my advisor Gennady Lyubeznik for suggesting this problem to me, as well as the referees for their helpful comments.

2. Preliminaries

In this section we have collected the ingredients of the main algorithm of the paper. We define the canonical form of a constructible set, consider parametric Gröbner bases for Weyl algebras, give a description of Oaku's algorithm for computing the Bernstein–Sato polynomial, and finally discuss the rationality of the roots of the Bernstein–Sato polynomial.

2.1. CONSTRUCTIBLE SETS

We recall that a set is constructible iff it is a finite union of locally closed sets and a set is locally closed iff it is the difference of two closed sets.

LEMMA 2.1. *Let C be a constructible subset of a k-variety X. Then C may be presented uniquely as a disjoint union $\bigcup_{i=1}^{m}(V_i' \setminus V_i'')$, where for all i the sets V_i' and V_i'' are closed, $V_1' \supset V_1'' \supset V_2' \supset V_2'' \supset \cdots \supset V_m' \supset V_m''$ and no two sets in this chain have an irreducible component in common. We call it a canonical presentation of C as a union of locally closed subsets.*

PROOF. Let $d(C)$ be the maximal dimension of an irreducible component in $\bar{C}$. Let $V_1' = \bar{C}$ and $V_1'' = \overline{V_1' \setminus C}$ and let $C' = C \cap V_1''$. Note that $d(C') < d(C)$ and we may assume by induction on d that the chain $V_2' \supset V_2'' \supset \cdots \supset V_m' \supset V_m''$ such that $C' = \bigcup_{i=2}^{m}(V_i' \setminus V_i'')$ exists and is unique. Then $V_1' \supset V_1'' \supset V_2' \supset V_2'' \supset \cdots \supset V_m' \supset V_m''$ is the unique chain for C, which satisfies the condition in the statement.

REMARK 2.2. There is an algorithmic way for constructing such a presentation, starting with C presented as a union of nonempty sets $W_\alpha \setminus (W_\alpha^{(1)} \cup \cdots \cup W_\alpha^{(h_\alpha)})$, where W_α and $W_\alpha^{(i)}$ are closed irreducible subsets and $W_\alpha \supset W_\alpha^{(i)}$ for all i. Let $d(C) = \max_\alpha \dim W_\alpha$ (which agrees with the definition in the proof of the theorem).

Let V_1' be the union of all maximal elements in the set $\{W_\alpha\}$ and V_1'' be the union of all $W_\alpha^{(i)}$ that are minimal with the following property: there is a set of pairs $\{(\alpha_j, i_j)\}_{j=1}^{l}$ such that W_{α_1} is a component of V_1', $W_{\alpha_l}^{(i_l)} = W_\alpha^{(i)}$ and $W_{\alpha_j}^{(i_j)} \supset W_{\alpha_{j-1}}$ for all $j = 2, \ldots, l$. Now $d(C \setminus (V_1' \setminus V_1''))$ is less than $d(C)$, therefore, we may assume again by induction on d that we are able to construct the rest of V_i' and V_i''.

LEMMA 2.3. *Let X be a variety and $f : X \to Y$ a map into any finite set Y. Then $f^{-1}(y)$ is constructible for every $y \in Y$ iff for every closed irreducible subvariety $X' \subset X$ there is an open $U \subset X'$ such that $f|_U$ is a constant function.*

PROOF. Assume the second part holds. Take any $y \in Y$ and let us prove that $Z = f^{-1}(y)$ is constructible. Let $n = \dim X$ and assume the lemma is proved for dimensions less then n. First of all, since X is a finite union of its irreducible components, and a subset of X is constructible iff its intersection with every irreducible component of X is, we may proceed assuming that X is irreducible. Let U be an open subset of X such that $f(u) = y'$ for all $u \in U$. There are two possibilities:

(i) if $y' \neq y$ then $Z \subset X \setminus U$, which has dimension less than n and, therefore, Z is constructible by the induction assumption applied to the map $f|_{X \setminus U} : X \setminus U \to Y$.

(ii) in case $y = y'$ the set $(Z \setminus U) \subset (X \setminus U)$ is constructible by the induction assumption again, hence so is $Z = U \cup (Z \setminus U)$.

It remains to check the case $\dim X = 0$, in which $f^{-1}(y)$ is a finite set of points and is certainly constructible.

Conversely, assume that $f^{-1}(y)$ is constructible for every $y \in Y$. Let $X' \subset X$ be a closed irreducible subvariety. Then $X' = \bigcup_{y \in Y} X_y'$, where $X_y' = (f^{-1}(y) \cap X')$, and, since Y is a finite set and X' is irreducible, there exist y such that the closure of X_y' is equal to X'. But X_y' is constructible, hence it contains a nonempty open subset of X'. □

2.2. parametric Gröbner bases

Here we describe an approach to computing parametric Gröbner bases in Weyl algebras. A good source on computing Gröbner bases in non-commutative algebras is Kandri-Rody and Weispfenning (1990). For a discussion of parametric Gröbner bases, which leads to the notion of comprehensive Gröbner bases, (see Weispfenning, 1992) for the commutative case and (Kredel and Weispfenning, 1991) for the case of solvable algebras. However, everything that is needed for this paper is stated and proved in this section.

Let $C = k[\bar{a}]$ ($\bar{a} = \{a_1, \ldots, a_m\}$) be the ring of parameters and $R = C\langle \bar{y}, \bar{x}, \bar{\partial}\rangle$ be the ring of non-commutative polynomials in $\bar{y} = \{y_1, \ldots, y_l\}$, $\bar{x} = \{x_1, \ldots, x_n\}$ and $\bar{\partial} = \{\partial_1, \ldots, \partial_n\}$ with coefficients in C, where $\bar{x}$ and $\bar{\partial}$ satisfy the same relations as in a Weyl algebra and $\bar{y}$ is contained in the center of R.

Definition 2.4. For a prime P in C, we shall call the natural map $C \to k(P)$ as well as the induced map $R = C\langle \bar{y}, \bar{x}, \bar{\partial}\rangle \to k(P)\langle \bar{y}, \bar{x}, \bar{\partial}\rangle$, where $k(P)$ is the residue field at P, the *specialization* at the point P and denote both maps by σ_P.

The next result is similar to Oaku's Proposition 7 in Oaku (1997).

Let $<$ be an order on monomials in $\bar{a}$, $\bar{y}$, $\bar{x}$ and $\bar{\partial}$ such that every a_i is $\ll$ than any of x_j, y_j or ∂_j (i.e. the order $<$ eliminates x_j, y_j and ∂_j). Assume G is a finite Gröbner basis of an ideal I of R, then we claim that $\sigma_P(G) = \{\sigma_P(g) \mid g \in G\}$ is a Gröbner basis of $\sigma_P(I)$ in $\sigma_P(R)$ for "almost" every $P \in \operatorname{Spec} C$.

In order to make this statement precise, we need to make some definitions. For a polynomial f let $inM(f)$ be the initial monomial $inC(f)$ the initial coefficient such that $in(f) = inC(f) \cdot inM(f)$ the initial term of f. Also for $f \in R$ let $inM_*(f) \in \langle \bar{y}, \bar{x}, \bar{\partial}\rangle$ and $inC_*(f) \in C$ be the initial monomial and the initial coefficient of f viewed as a polynomial in x, y, ∂ with coefficients in C with respect to $\prec$, the restriction of $<$ to $\langle \bar{y}, \bar{x}, \bar{\partial}\rangle$.

One obvious observation is that a specialization $\sigma_P : (R, <) \to (\sigma_P(R), \prec)$ preserves the order.

Lemma 2.5. *Let Q be an ideal contained in I and let $h = \prod_{g \in G \setminus Q} inC_*(g) \in C$. Then $\sigma_P(G \setminus Q)$ is a Gröbner basis of $\sigma_P(I)$ for every prime $P \supset Q$ not containing h.*

Proof. Notice that if any $g \in G \setminus Q$ has $inC_*(g) \in Q$ then the statement of the lemma becomes trivial.

Assume $inC_*(g) \notin Q$ for all $g \in G \setminus Q$. Consider any prime $P \supset Q$ not containing h. Take a polynomial $f' = \sum_{g \in G \setminus Q} \frac{[\alpha_g]}{[\beta_g]} \sigma_P(g)$ in the ideal of $\sigma_P(R)$ generated by $\sigma_P(G)$, where $\alpha_g, \beta_g \in C$, $\beta_g \notin P$ and $[\ldots]$ stands for an equivalence class in C/P. Set $\gamma = \prod_{g \in G} \beta_g$ then $(\sum_{g \in G \setminus Q} \frac{\gamma \alpha_g}{\beta_g} g)$ is in the ideal I of R. Let f be the latter sum where all terms with coefficients in Q are set to zero. Then $f' = \frac{1}{[\gamma]} \sigma_P(f)$ and $inM_*(f) = inM(f')$. We have $inM(g) | inM(f)$ for some $g \in G \setminus Q$, which means that $inM_*(g) | inM_*(f)$. Now, $inM(\sigma_P(g)) = inM_*(g)$, because $inC_*(g) \notin P$. Thus $inM(\sigma_P(g)) | inM(\sigma_P(f))$, which proves that $\sigma_P(G)$ is a Gröbner basis.

Remark 2.6. (i) The statement of the lemma is true for reduced Gröbner bases as well. The proof works almost verbatim.

(ii) Clearly, $inC_*(g) \notin Q$ for all $g \in G \setminus Q$. Thus if Q is prime, then $h \notin Q$, hence the set of primes containing Q but not containing h is nonempty.

The lemma leads to the following:

ALGORITHM 2.7.

Input:	F' :	a finite set of generators for a prime ideal $Q \subset C$.
	F :	a finite set of generators of a left ideal $I \subset R$ containing Q,
Output:	G :	a (reduced) Gröbner basis in R with respect to $<$,
	h :	an *exceptional polynomial* in $C \setminus Q$, such that for any $P \in \operatorname{Spec}(k[a_1, \ldots, a_m])$, $P \supset Q$ and $h \notin P$ the ideal $\sigma_P(I) \subset \sigma_P(R)$ has a $\sigma_P(G)$ as a (reduced) Gröbner basis with respect to $\prec$.

(1) Compute a Gröbner basis G of $I + QR$ (which is generated by $F \cup F'$).
(2) Return G and $h = \prod_{g \in G \setminus Q} inC_*(g)$.

REMARK 2.8. If all polynomials in F' and all C-coefficients of all elements of F are homogeneous, then so is the exceptional polynomial h, because all operations preserve homogeneity.

2.3. Oaku's algorithm

The original algorithm of Oaku for computing the Bernstein–Sato polynomial appeared in Oaku (1997). However there exist several modifications of the algorithm (see, Saito *et al.*, 2000, for example). For our needs a version of the algorithm described in Walther (1999) will be utilized.

Let $f \in R_n(k)$. Denote by $\operatorname{Ann} f^s$ the ideal of all elements in $A_n(k)[s]$ annihilating f^s. The following algorithm is Algorithm 4.4. from Walther (1999) with $L = (\partial_1, \ldots, \partial_n)$.

ALGORITHM 2.9.

Input:	f:	a polynomial in $R_n(k)$,
Output:	$\{P'_j\}$:	generators of $\operatorname{Ann} f^s$

(1) Set $Q = \left\{\partial_i + \frac{df}{dx_i}\partial_t, t - f\right\}$.
(2) Introduce new variables y_1 and y_2 and the weight w such that $w(t) = w(y_1) = 1$, $w(\partial_t) = w(y_2) = -1$, $w(x_i) = w(\partial_i) = 0$. Homogenize all $q_i \in Q$ $(i = 1, \ldots, n+1)$ using y_1 with respect to the weight w. Denote the homogenized elements q_i^h.
(3) Compute a Gröbner basis for the ideal generated by $q_1^h, \ldots, q_{n+1}^h$, $1 - y_1 y_2$ in $A_{n+1}[y_1, y_2]$ with respect to an order eliminating y_1, y_2.
(4) Select the operators $\{P_j\}_1^b$ in this basis which do not contain y_1, y_2.
(5) For each P_j, if $w(P_j) > 0$ then replace P_j by $P'_j = \partial_t^{w(P_j)} P_j$ else replace P_j by $P'_j = t^{-w(P_j)} P_j$.
(6) Return the operators $\{P'_j\}_1^b$.

The following is Algorithm 4.6 in Walther (1999).

Algorithm 2.10.

Input:	f :	a polynomial in $R_n(k)$,
Output:	$b_f(s)$	the Bernstein–Sato polynomial of f.

(1) Determine $\mathrm{Ann} f^s$ following Algorithm 2.9.
(2) Find a reduced Gröbner basis for the ideal $\mathrm{Ann} f^s + A_n[s] \cdot f$ using an order that eliminates x and ∂.
(3) Return the unique element in the basis contained in $k[s]$.

2.4. rationality of the roots

The first paper that discussed the rationality of the roots of the Bernstein–Sato polynomials was Malgrange (1975) by B. Malgrange. Using resolution of singularities, Kashiwara in Kashiwara (1976/77) proved that the roots of local Bernstein–Sato polynomials are rational when $k = \mathbb{C}$. In Mebkhout and Narváez-Macarro (1991, Proposition 4.2.1) it is proved that the Bernstein–Sato polynomial $b_f(s)$ is the lowest common multiple of the local Bernstein–Sato polynomials. Hence the roots of $b_f(s)$ are rational if $k = \mathbb{C}$.

In particular, it follows that $b_f(s) \in \mathbb{Q}[s]$. The fact that the roots are rational for every k is well-known to experts, but we have not been able to find a published proof, so we prove it in the next proposition.

Proposition 2.11. *Let k be a field, char $k = 0$. Then for every $f \in R_n(k)$ the roots of the Bernstein–Sato polynomial $b_f(s)$ are rational.*

Proof. The crucial fact is that if $K \subset k$ is a subfield containing all the coefficients of f, then the coefficients of $b_f(s)$ computed over k belong to K. This is because upon examining every step of Oaku's algorithm one sees that all calculations are done in K. Let K be a finite extension of $\mathbb{Q}$ containing the coefficients of f. Since one can embed K into $\mathbb{C}$, the Bernstein–Sato polynomial of f over K is the same as over $\mathbb{C}$. Now we are done by Kashiwara's result in conjunction with Mebkhout and Narváez-Macarro (1991).

3. The Main Results

Consider $\mathbb{P}(n, d; k)$ with the coordinate ring $C = k[\bar{a}]$, where $\bar{a} = \{a_\alpha : |\alpha| \le d\}$. Let $f = \sum_{|\alpha| \le d} a_\alpha x^\alpha$.

Definition 3.1. Let $b(s) \in B(n, d)$. The set $S(b(s), k) \subset \mathbb{P}(n, d; k)$ is defined as the set of all the points $P \in \mathbb{P}(n, d; k)$ such that $b_{\sigma_P(f)}(s) = b(s)$. (We view points in $\mathbb{P}(n, d; k)$ as homogeneous primes in C. See Definition 2.4 for $\sigma_P(f)$.)

Lyubeznik's proof that $B(n, d)$ is finite can be summarized as follows. Let the space of parameters $X = \mathrm{Spec}\, A$ be an irreducible variety with A a quotient ring of $\mathbb{Q}[\bar{a}]$. We consider the Bernstein–Sato polynomial $b(s)$ of polynomial f from the beginning of this section seen as a polynomial over the field of fractions of A. Using the rationality of $b(s)$ and clearing the denominators, we obtain a functional equation

$$Q(a, x, \partial, s) f^{s+1} = h(a) b(s) f^s.$$

This shows that outside the zeroes of $h \in A$, the Bernstein–Sato polynomial divides $b(s)$. Hence there are only finitely many possibilities for $b(s)$ outside the zeros of h, while the set of zeros of h has a smaller dimension than X. This allows an induction argument for the finiteness of $B(n,d)$. But to prove constructibility one needs to find such h that, outside its zeroes, the Bernstein–Sato polynomial equals $b(s)$. This is the idea of the proof.

Let Q be a homogeneous prime in C. Then $\sigma_Q(f)$ is a polynomial with coefficients in a field, hence $b_{f_Q}(s)$ may be computed. What would happen if we run Algorithm 2.10 trying to compute $b_{f_Q}(s)$ "lifting from $k(Q)$, the fraction field of C/Q, to C" every single step of the algorithm? Notice that $\sigma_Q : C \to k(Q)$ has C/Q as its image. Since the steps of the algorithm that do not involve Gröbner bases computation do not involve division either, we have to worry only about the two steps that deal with Gröbner bases. Suppose for these two steps we used Algorithm 2.7 with F' generating Q, in particular we obtained the exceptional polynomials h_1 and h_2, both in C. Set $h = h_1 h_2 \in C$, then the output, which is going to be $b_{\sigma_Q(f)}(s)$, is also the Bernstein–Sato polynomial of $\sigma_P(f)$ for every $P \supset Q$ such that $h \notin P$. Thus we have

Algorithm 3.2.

Input:	f:	a polynomial in $R_n(C)$,
	F':	generators of a homogeneous prime ideal Q,
Output:	$b(s)$:	a polynomial in $\mathbb{Q}[s]$,
	H:	generators of a homogeneous ideal in C such that $b(s) = b_{\sigma_P(f)}(s)$ for every point $P \in V' \backslash V'' \neq 0$, where $V' = V(Q)$ and $V'' = V(H)$ $(V'' \subset V' \subset \mathbb{P}(n,d;k))$.

(1) Compute the polynomial $b(s)$ and the exceptional polynomial h as described above.
(2) Return $b(s)$ and $\{h\} \cup F'$.

Proposition 3.3. *If we consider $C' = C \otimes_k k'$ and $f \otimes_k 1 \in R_n(C')$, where k' is an extension of k, then $b(s)$ is the Bernstein–Sato polynomial for any point in the set $(V' \otimes_{Spec\ k} Spec\ k') \setminus (V'' \otimes_{Spec\ k} Spec\ k')$.*

Proof. Let Q be as above, then QC' may not be prime anymore. Nevertheless, assume the computation above was done for $f \otimes 1 \in R_n(C')$ and QC' as input. This computation "stays within k", i.e. no operation introduces an element outside the old ring. The output of the algorithm would be the same as before, and we claim that for every prime $P' \supset QC'$ not containing $h \otimes 1$ the Bernstein–Sato polynomial of $\sigma_P(f \otimes 1)$ is equal to $b_{\sigma_Q(f)}(s)$. This is guaranteed by Lemma 2.5. □

Remark 3.4. One can prove easily that $(V' \otimes_{\mathrm{Spec}\ k} \mathrm{Spec}\ k') \setminus (V'' \otimes_{\mathrm{Spec}\ k} \mathrm{Spec}\ k')$ is nonempty (although we will not use this fact in the sequel) by showing that:

(i) QC' is the intersection of its associated primes $\{Q_i\} \subset \mathrm{Spec}\ C'$,
(ii) no Q_i contains $h \otimes 1$, for otherwise $Q_i \cap C = Q$ contains h.

The next theorem gives an affirmative answer to Lyubeznik's question about the constructibility of the set $S(b(s),k)$ of Definition 3.1.

Theorem 3.5. *The set $S(b(s), k)$ is constructible for every $b(s)$.*

Proof. The proof follows from the algorithm. For the function $\phi : \mathbb{P}(n, d; k) \to B(n, d)$, $\phi(P) = b_{\sigma_P(f)}(s)$ the following is true. For every projective $V' \subset \mathbb{P}(n, d; k)$ there is an open set $U = V' \setminus V'' \subset V'$ such that $f|_U$ is a constant function. Therefore we may apply Lemma 2.3. □

Algorithm 3.2 leads to the main algorithm of the paper.

Algorithm 3.6. *Input: $n, d \in \mathbb{N}$.*
Output: The set of pairs $L = \{(b(s), S(b(s)))|\, b(s) \in B(n, d)\}$, where $S(b(s)) = S(b(s), \mathbb{Q}) \subset \mathbb{P}(n, d; \mathbb{Q})$.

(1) Set $L := \emptyset$, $f := \sum_{|\alpha| \leq d} a_\alpha x^\alpha$.
(2) Define the recursive procedure $\mathbf{BSP}(Q)$, where $Q \in \mathrm{Spec}(\mathbb{Q}[\bar{a}])$.

```
BSP(Q) := {
 Apply Algorithm 3.2 to V(Q) and f
          to get an ideal I in C and b(s) ∈ Q[s];
 IF there is a pair (b(s), S) ∈ L
   THEN replace it by (b(s), S ∪ (V(Q) \ V(I)))
   ELSE L := L ∪ {(b(s), V(Q) \ V(I))};
 IF V(I) ≠ ∅ THEN {
   Find the minimal primes {Q_i} associated to I;
   FOR each Q_i DO BSP(Q_i) ;
 }
}
```

(3) Run BSP(0).

Remark 3.7. This algorithm returns some presentations for constructible sets $S(b(s),\mathbb{Q})$, the canonical presentations for which may be obtained by using the algorithm discussed in Remark 2.2.

Corollary 3.8. *The set $S(b(s), k)$ is defined over $\mathbb{Q}$, i.e. there exist ideals $I_i \subset \mathbb{Q}[\bar{a}]$ and $J_i \subset \mathbb{Q}[\bar{a}]$ $(i = 1, \ldots, m)$ such that for any field k*

$$S(b(s), k) = \bigcup_i (V_i' \setminus V_i''),$$

where $V_i' = V(k[\bar{a}]I_i)$ is the zero set of the extension of I_i and $V_i'' = V(k[\bar{a}]J_i)$ is the zero set of the extension of J_i.

Proof. Since the core part of algorithm above is Algorithm 3.2, the statement of the corollary follows from Proposition 3.3 applied to the extension k of $\mathbb{Q}$. □

Remark 3.9. Given a polynomial with parameters one can use a similar approach to compute the stratification of the parameter space corresponding to the set of all possible Bernstein–Sato polynomials (see Examples 4.3 and 4.4).

The annihilators $\mathrm{Ann}(f^s)$ are computed using Algorithm 2.9 and the same technique as in the algorithm above. The output is a set of pairs $\{(I_i, V_i)\}$, where I_i are the ideals in $A_n(k)[\bar{a}][s]$ and V_i are locally closed sets, such that for any polynomial f with coefficients in k that corresponds to a point $P \in V_i$ the ideal $\mathrm{Ann}(f^s)$ equals $\sigma_P(I_i)$, the ideal I_i specialized to P.

After doing the above steps, the real life algorithm that produces Example 4.5 compresses its output in the following way. If (I_i, V_i) and (I_j, V_j) are two different pairs such that $\sigma_P(I_i) = \sigma_P(I_j)$ for all $P \in V_j$ then these two are replaced by the pair $(I_i, V_i \cup V_j)$.

Remark 3.10. The stratification of the parameter space constructed by such computation is not unique. This is so because the annihilators, as opposed to Bernstein–Sato polynomials, depend on the parameters, making it possible to slice the space of parameters in many ways.

4. Examples

Our algorithms have been implemented as scripts written in the Macaulay 2 programming language (see Grayson and Stillman). In this section we give some examples of actual computations and discuss possible uses of the results of computation.

Example 4.1. If $n = 2$ and $d = 2$ then

$$f = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00},$$

so $P(2, 2; k)$ is the set of the k-rational points of the projective space $\mathbb{P}(2, 2; k) = \mathbb{P}^5_k$ with the homogeneous coordinate ring $k[a_{ij}]$, $i, j = 0, 1, 2$. It takes our program less than 20 minutes on 300 MHz Pentium-II machine to produce

$$B(2,2) = \left\{1,\ s+1,\ (s+1)^2,\ (s+1)\left(s+\frac{1}{2}\right)\right\}$$

and give a description of the corresponding constructible sets of polynomials from $B(2, 2)$ which is essentially equivalent to the following:

- $b_f(s) = 1$ iff $f \in V_1 = V_1' \setminus V_1''$, where $V_1' = V(a_{1,1}, a_{0,1}, a_{0,2}, a_{1,0}, a_{2,0})$, while $V_1'' = V(a_{0,0})$,
- $b_f(s) = s + 1$ iff $f \in V_2 = (V_2' \setminus V_2'') \cup (V_3' \setminus V_3'')$, where $V_2' = V(0)$, $V_2'' = V(\gamma_1)$, $V_3' = V(\gamma_2, \gamma_3, \gamma_4)$, $V_3'' = V(\gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8)$,
- $b_f(s) = (s + 1)^2$ iff $f \in V_4' \setminus V_4''$, where $V_4' = V(\gamma_1)$, $V_4'' = V(\gamma_2, \gamma_3, \gamma_4)$,
- $b_f(s) = (s + 1)\left(s + \frac{1}{2}\right)$ iff $f \in V_5' \setminus V_5''$, where $V_5' = V(\gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8)$, while $V_5'' = V(a_{1,1}, a_{0,1}, a_{0,2}, a_{1,0}, a_{2,0})$,

where γ_i may be looked up in this list:

$$\begin{aligned}
\gamma_1 &= a_{0,2}a_{1,0}^2 - a_{0,1}a_{1,0}a_{1,1} + a_{0,0}a_{1,1}^2 + a_{0,1}^2a_{2,0} - 4a_{0,0}a_{0,2}a_{2,0},\\
\gamma_2 &= 2a_{0,2}a_{1,0} - a_{0,1}a_{1,1},\\
\gamma_3 &= a_{1,0}a_{1,1} - 2a_{0,1}a_{2,0},\\
\gamma_4 &= a_{1,1}^2 - 4a_{0,2}a_{2,0},\\
\gamma_5 &= 2a_{0,2}a_{1,0} - a_{0,1}a_{1,1},\\
\gamma_6 &= a_{0,1}^2 - 4a_{0,0}a_{0,2},\\
\gamma_7 &= a_{0,1}a_{1,0} - 2a_{0,0}a_{1,1},\\
\gamma_8 &= a_{1,0}^2 - 4a_{0,0}a_{2,0}.
\end{aligned}$$

It is not hard to see that this computation agrees with the well-known result that $b_f(s) = 1$ iff f is constant, $b_f(s) = s + 1$ iff f is non-constant and non-singular, and $b_f(s) = (s+1)^2$ $\bigl($resp. $-b_f(s) = (s+1)(s+\frac{1}{2})\bigr)$ iff f can be reduced to xy (resp. x^2) by a linear change of variables.

Example 4.2. If $n = 2$ and $d = 3$ then

$$\begin{aligned} f = {} & a_{3,0}x^3 + a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3 \\ & + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + a_{1,0}x + a_{0,1}y + a_{0,0}, \end{aligned}$$

so $P(2,3;k)$ is the set of the k-rational points of $\mathbb{P}(2,3;k) = \mathbb{P}^9_k$ with the homogeneous coordinate ring that involves 10 variables. Our program exhausts all available memory, 128 Mb, of the computer after about 3 hours and stops without producing an answer. However, a somewhat creative use of our program enables us to give a complete list of all the elements of $B(2,3)$ (but not the explicit descriptions of the constructible sets corresponding to each element of $B(2,3)$):

Since for any nonsingular polynomial its Bernstein–Sato polynomial is equal to $s + 1$, it remains to consider the case where our $f \in \mathbb{P}(2,3;k)$ possesses a singularity at some point (x_0, y_0). Keeping in mind that the Bernstein–Sato polynomial is stable under any linear substitution of variables, we may get rid of its linear part via the substitution $x \mapsto x - x_0$, $y \mapsto y - y_0$, i.e. f takes the form

$$f = (ax^3 + bx^2y + cxy^2 + dy^3) + (a'x^2 + b'xy + c'y^2).$$

Now it is easy to see that by homogeneous linear transformation the quadratic part may be shaped to one of the forms 0, xy, x^2. Therefore it is enough to compute the Bernstein–Sato polynomial for the following polynomials:

$$\begin{aligned} f_1 &= ax^3 + bx^2y + cxy^2 + dy^3, \\ f_2 &= (ax^3 + bx^2y + cxy^2 + dy^3) + xy, \\ f_3 &= (ax^3 + bx^2y + cxy^2 + dy^3) + x^2. \end{aligned}$$

Our program returns the complete sets of possible Bernstein–Sato polynomials for f_1 in 22 minutes, for f_2 in 16 minutes and for f_3 in 21 minutes. Of course, in each of the three cases our program produces an explicit description of the corresponding constructible set in $\mathbb{A}^4_k$ (each f_i contains four indeterminate coefficients) for each element $b(s) \in B_{f_i}$. We omit these and list only the Bernstein–Sato polynomials:

$$\begin{aligned} B_{f_1} = \Bigl\{ & (s+1)^2\left(s+\frac{2}{3}\right)\left(s+\frac{4}{3}\right), \\ & (s+1)^2\left(s+\frac{1}{2}\right), \\ & (s+1)\left(s+\frac{2}{3}\right)\left(s+\frac{1}{3}\right)\Bigr\}; \\ B_{f_2} = \{ & (s+1)^2 \}; \\ B_{f_3} = \Bigl\{ & (s+1)\left(s+\frac{7}{6}\right)\left(s+\frac{5}{6}\right), \\ & (s+1)^2\left(s+\frac{3}{4}\right)\left(s+\frac{5}{4}\right), \end{aligned}$$

$$(s+1)^2\left(s+\frac{1}{2}\right),$$
$$(s+1)\left(s+\frac{1}{2}\right)\bigg\}.$$

Thus

$$\begin{aligned}B(2,3)=\bigg\{&(s+1)^2\left(s+\frac{2}{3}\right)\left(s+\frac{4}{3}\right),\\&(s+1)^2\left(s+\frac{1}{2}\right),\\&(s+1)\left(s+\frac{2}{3}\right)\left(s+\frac{1}{3}\right),\\&(s+1)^2,\\&(s+1)\left(s+\frac{7}{6}\right)\left(s+\frac{5}{6}\right),\\&(s+1)^2\left(s+\frac{3}{4}\right)\left(s+\frac{5}{4}\right),\\&(s+1)\left(s+\frac{1}{2}\right),\\&s+1,\\&1\bigg\}.\end{aligned}$$

The efficiency of the algorithm and the current efficiency of computer hardware and software obstruct us from getting a complete description of the constructible sets that correspond to the polynomials above.

Here are a couple of examples of the computation for polynomials with parameters.

Example 4.3. Let $f=x^3+ax+b+cy^4+y^2$, then

- $b(s)=(s+1)$ for $V(0)\setminus(V\left(a^3c^2+\frac{27}{4}b^2c^2-\frac{27}{8}bc+\frac{27}{64}\right)\cup V(4a^3+27b^2))$,
- $b(s)=(s+1)^2$ for $(V\left(a^3c^2+\frac{27}{4}b^2c^2-\frac{27}{8}bc+\frac{27}{64}\right)\cup V(4a^3+27b^2))\setminus(V(a,4bc-1)\cup V(a,b))$,
- $b(s)=(s+1)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)$ for $V(a,4bc-1)\cup V(a,b)$.

(Computation time = 1 min 45 sec)

Example 4.4. Let $f=x^2+axy+by^2+z^3+cx^4$, then
- $b(s)=(s+1)\left(s+\frac{4}{3}\right)\left(s+\frac{5}{3}\right)$ for $V(0)\setminus V(a^2-4b)$,
- $b(s)=(s+1)\left(s+\frac{4}{3}\right)\left(s+\frac{5}{3}\right)\left(s+\frac{13}{12}\right)\left(s+\frac{17}{12}\right)\left(s+\frac{19}{12}\right)\left(s+\frac{23}{12}\right)$ for $V(a^2-4b)\setminus V(c,a^2-4b)$,
- $b(s)=(s+1)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)$ for $V(a,b)\cup V(c,a^2-4b)$.

(Computation time = 6 minutes)

Using a technique similar to that for computing Bernstein–Sato polynomials, we constructed an algorithm for computing of $\mathrm{Ann} f^s$, the annihilator ideal of f^s in $A_n(k)[s]$, for all $f\in P(n,d;k)$. By this we mean an explicit subdivision of $\mathbb{P}(n,d;k)$ into a finite union of constructible subsets and for each such subset V, an explicit finite set of elements $\beta_1, \beta_2, \ldots \in A_n(k)[a_{i_1\ldots i_n}][s]$ with $i_1 + \cdots + i_n \leq d$, such that $\mathrm{Ann}(f^s) = (\beta'_1, \beta'_2, \ldots)$ for every $f \in V$, where β'_i is the image of β_i under the specialization of the $a_{i_1\ldots i_n}$ to the corresponding coefficients of f.

Example 4.5. To make the results obtained for $P(2,2;k)$ compact we need the following polynomials:

$$\begin{aligned}
\beta_1 &= a_{1,1}x_1\partial_1 + 2a_{0,2}x_2\partial_1 - 2a_{2,0}x_1\partial_2 - a_{1,1}x_2\partial_2 + a_{0,1}\partial_1 - a_{1,0}\partial_2,\\
\beta_2 &= a_{1,1}a_{2,0}x_1^2\partial_1 + a_{1,1}^2x_1x_2\partial_1 + a_{0,2}a_{1,1}x_2^2\partial_1 - 2a_{2,0}^2x_1^2\partial_2 - 2a_{1,1}a_{2,0}x_1x_2\partial_2\\
&\quad -2a_{0,2}a_{2,0}x_2^2\partial_2 - a_{1,1}^2sx_2 + 4a_{0,2}a_{2,0}sx_2 + a_{1,0}a_{1,1}x_1\partial_1 + a_{0,1}a_{1,1}x_2\partial_1\\
&\quad -2a_{1,0}a_{2,0}x_1\partial_2 - 2a_{0,1}a_{2,0}x_2\partial_2 - a_{1,0}a_{1,1}s + 2a_{0,1}a_{2,0}s + a_{0,0}a_{1,1}\partial_1 - 2a_{0,0}a_{2,0}\partial_2,\\
\beta_3 &= a_{2,0}x_1^2\partial_2 + a_{1,1}x_1x_2\partial_2 + a_{0,2}x_2^2\partial_2 - a_{1,1}sx_1 - 2a_{0,2}sx_2 + a_{1,0}x_1\partial_2\\
&\quad +a_{0,1}x_2\partial_2 - a_{0,1}s + a_{0,0}\partial_2,\\
\beta_4 &= a_{1,1}^2x_1\partial_1 - 4a_{0,2}a_{2,0}x_1\partial_1 + a_{1,1}^2x_2\partial_2 - 4a_{0,2}a_{2,0}x_2\partial_2 - 2a_{1,1}^2s\\
&\quad +8a_{0,2}a_{2,0}s - 2a_{0,2}a_{1,0}\partial_1 + a_{0,1}a_{1,1}\partial_1 + a_{1,0}a_{1,1}\partial_2 - 2a_{0,1}a_{2,0}\partial_2,\\
\beta_5 &= a_{1,1}\partial_1 - 2a_{2,0}\partial_2,\\
\beta_6 &= 2a_{2,0}x_1\partial_2 + a_{1,1}x_2\partial_2 - 2a_{1,1}s + a_{1,0}\partial_2,\\
\beta_7 &= \partial_1,\\
\beta_8 &= 2a_{0,2}x_2\partial_2 - 4a_{0,2}s + a_{0,1}\partial_2,\\
\beta_9 &= \partial_2,\\
\beta_{10} &= 2a_{2,0}x_1\partial_1 - 4a_{2,0}s + a_{1,0}\partial_1,\\
\beta_{11} &= a_{2,0}x_1^2\partial_1 - 2a_{2,0}sx_1 + a_{1,0}x_1\partial_1 - a_{1,0}s + a_{0,0}\partial_1,\\
\gamma_1 &= a_{0,2}a_{1,0}^2 - a_{0,1}a_{1,0}a_{1,1} + a_{0,0}a_{1,1}^2 + a_{0,1}^2a_{2,0} - 4a_{0,0}a_{0,2}a_{2,0},\\
\gamma_2 &= 2a_{0,2}a_{1,0} - a_{0,1}a_{1,1},\\
\gamma_3 &= a_{1,0}a_{1,1} - 2a_{0,1}a_{2,0},\\
\gamma_4 &= a_{1,1}^2 - 4a_{0,2}a_{2,0},\\
\gamma_5 &= a_{0,1}^2 - 4a_{0,0}a_{0,2},\\
\gamma_6 &= a_{0,1}a_{1,0} - 2a_{0,0}a_{1,1},\\
\gamma_7 &= a_{1,0}^2 - 4a_{0,0}a_{2,0}.
\end{aligned}$$

Here are all the possible annihilators together with their strata:

- $\mathrm{Ann}(f^s) = (\beta_1, \beta_2, \beta_3)$ for $f \in (V'_1 \setminus V''_1) \cup (V'_2 \setminus (V''_{2,1} \cup V''_{2,2}))$, where $V'_1 = V(0)$, $V''_1 = V(\gamma_1)$, $V'_2 = V(\gamma_2, \gamma_3, \gamma_4)$, $V''_{2,1} = V(a_{1,1}, a_{0,2}, a_{0,1})$ and $V''_{2,2} = V(\gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7)$;
- $\mathrm{Ann}(f^s) = (\beta_1, \beta_4)$ for $f \in V'_3 \setminus V''_3$, where $V'_3 = V(\gamma_1)$, while $V''_3 = V(\gamma_2, \gamma_3, \gamma_4)$;
- $\mathrm{Ann}(f^s) = (\beta_5, \beta_6)$ for $f \in V'_4 \setminus (V''_{4,1} \cup V''_{4,2})$ where $V'_4 = V(\gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7)$, $V''_{4,1} = V(a_{1,0}, a_{2,0}, a_{1,1}, \gamma_5)$ and $V''_{4,2} = (a_{1,1}, a_{0,2}, a_{0,1}, \gamma_7)$;
- $\mathrm{Ann}(f^s) = (\beta_7, \beta_8)$ for $f \in V'_5 \setminus V''_5$, where $V'_5 = V(a_{1,0}, a_{2,0}, a_{1,1}, \gamma_5)$, while $V''_5 = V(a_{1,1}, a_{0,1}, a_{0,2}, a_{1,0}, a_{2,0})$;
- $\mathrm{Ann}(f^s) = (\beta_9, \beta_{10})$ for $f \in V'_6 \setminus V''_6$, where $V'_6 = V(a_{1,1}, a_{0,2}, a_{0,1}, \gamma_7)$, while $V''_5 = V(a_{1,1}, a_{0,1}, a_{0,2}, a_{1,0}, a_{2,0})$;
- $\mathrm{Ann}(f^s) = (\beta_9, \beta_{11})$ for $f \in (V'_7 \setminus V''_7) \cup V'_8$, where $V'_7 = V(a_{1,1}, a_{0,2}, a_{0,1})$, $V''_7 = V(a_{1,1}, a_{0,2}, a_{0,1}, \gamma_7)$ and $V'_8 = V(a_{1,1}, a_{0,1}, a_{0,2}, a_{1,0}, a_{2,0})$;

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